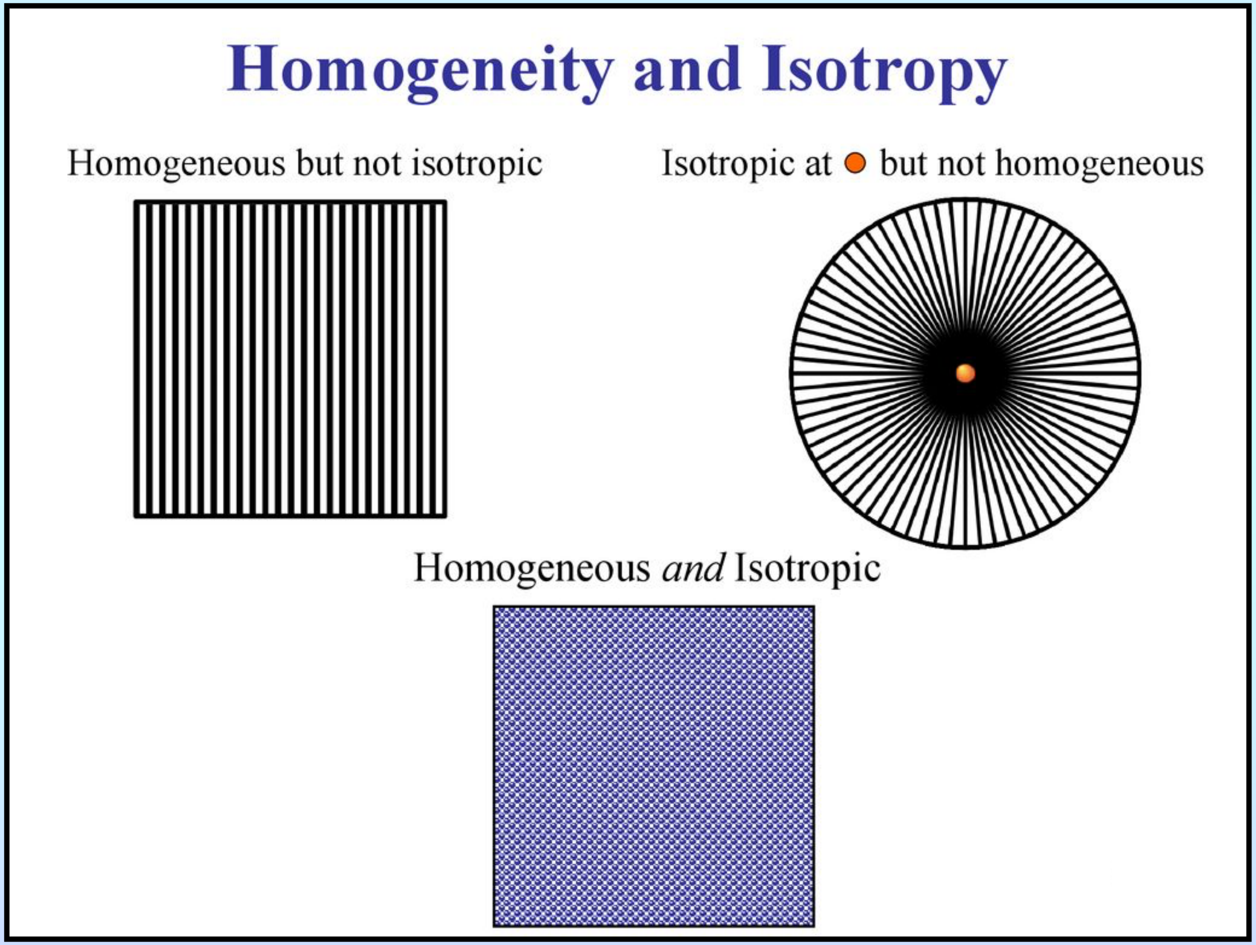
Basic Cosmological Assumption

- Cosmological Principle – the universe is isotropic and homogeneous
  - It appears the same in all directions and at all locations



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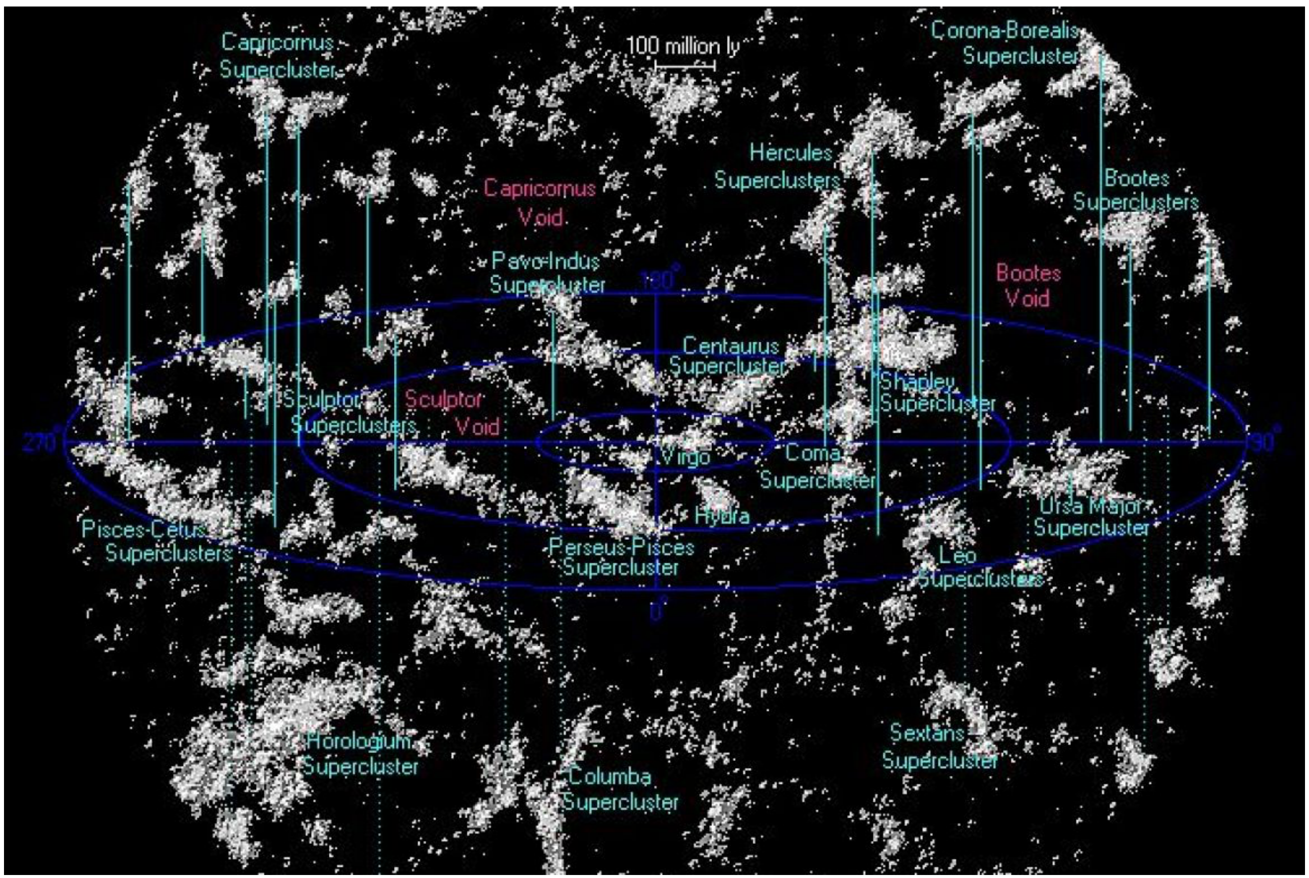

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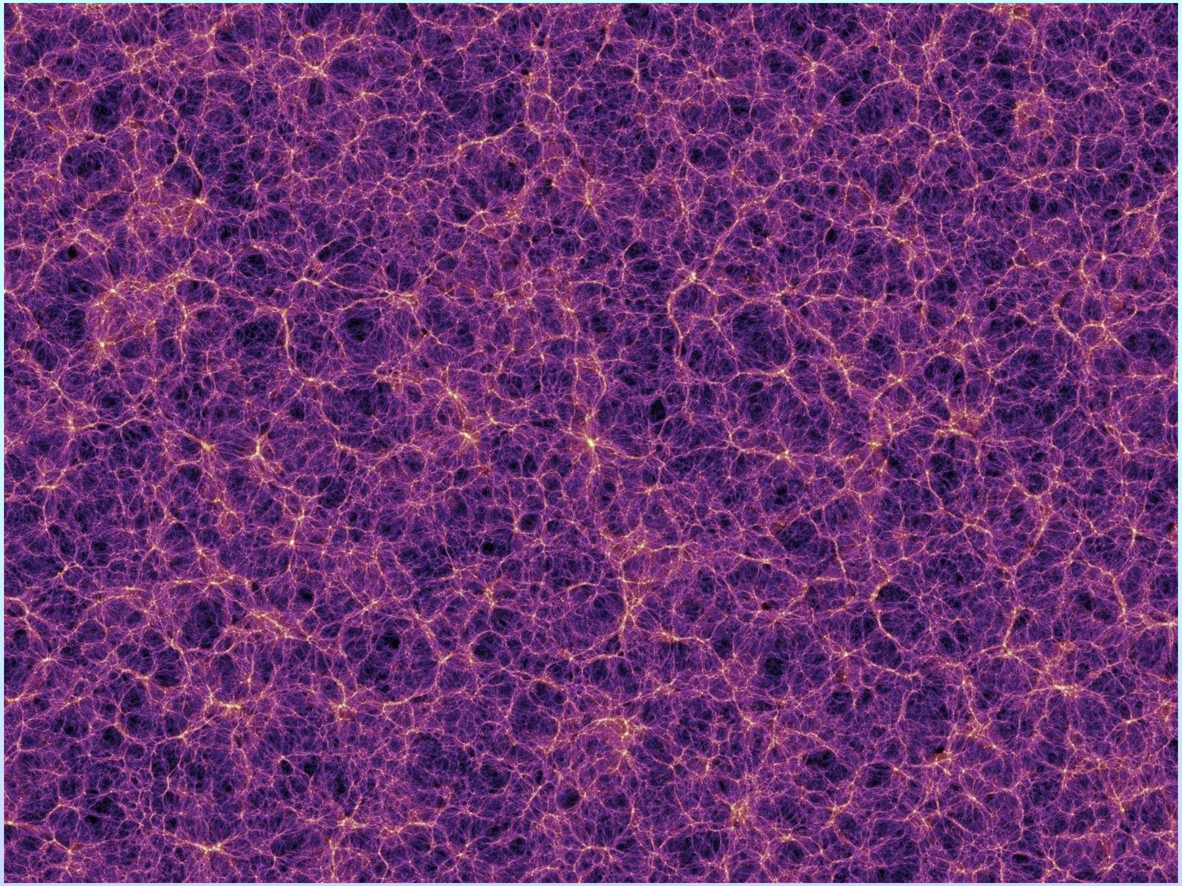
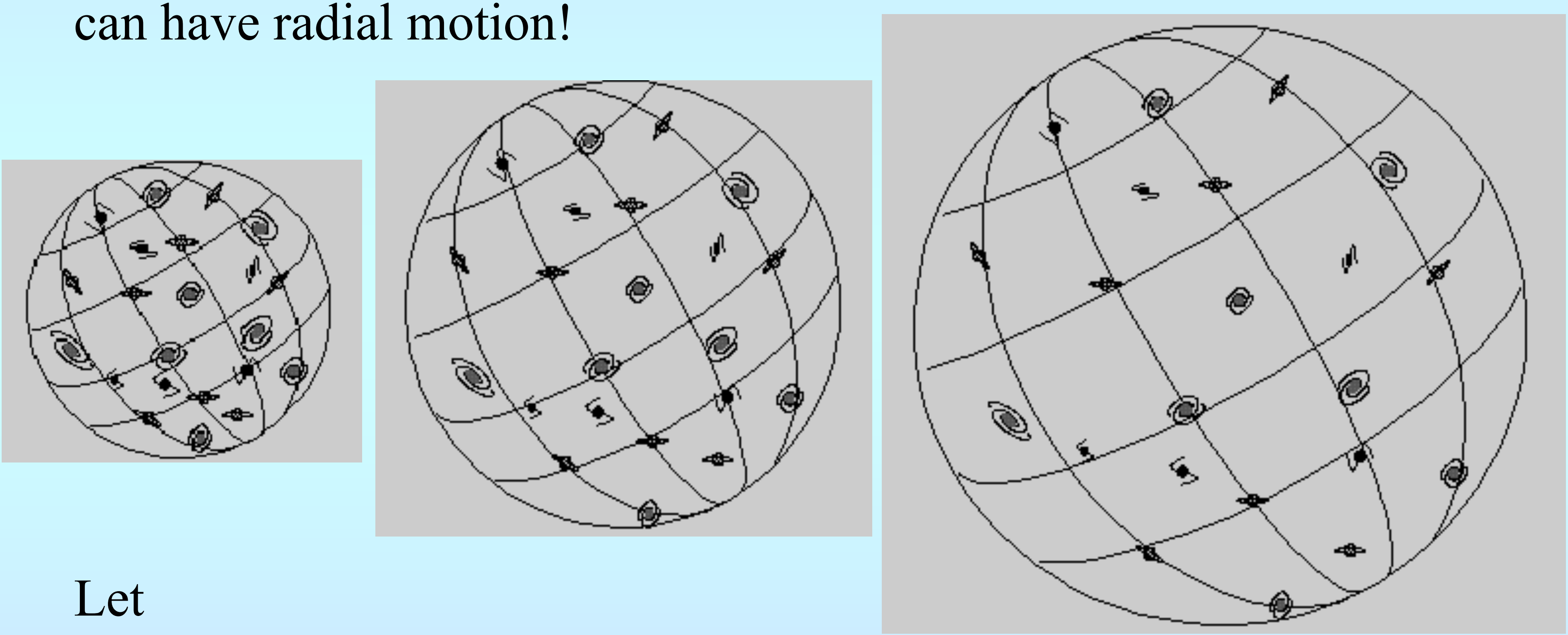


A Dynamic Universe

The cosmological principal requires that there be no rotation in the universe (i.e., it would mean having a preferred axis). But it doesn't mean that the entire universe must be static – the universe can have radial motion!



Let

u = the *co-moving* coordinates of an object

$a(t)$ = the scale factor of the universe

ℓ = the measurable distance to a galaxy

A Dynamic Universe

u = the *co-moving* coordinates of an object

$a(t)$ = the scale factor of the universe

ℓ = the measurable distance to a galaxy, i.e., $\ell = a\,u$

Looked at in this way, the galaxies are *not* moving: they are fixed in space. But the universe itself may be changing its size.

If the universe is indeed expanding, then the galaxy distances will appear to change. Specifically, the velocity we measure will be

$$v = \frac{d\ell}{dt} = \frac{d}{dt}(au) = \frac{da}{dt}u = \dot{a}u$$

Note that we cannot measure *a,* and we don't know *u* ahead of time. But in principal, we can measure ℓ. So let's substitute for *u*

$$v = \dot{a}u = \dot{a}\left(\frac{\ell}{a}\right) = \left(\frac{\dot{a}}{a}\right)\ell$$

A Dynamic Universe

$$v = \dot{a}u = \dot{a}\left(\frac{\ell}{a}\right) = \left(\frac{\dot{a}}{a}\right)\ell = H\ell$$

where we have defined $H(t) = \dot{a}/a$. Note that $H(t)$ has units of inverse time, and represents the fractional rate that the universe is changing its size. This is the Hubble parameter. The value of the Hubble parameter today is called the Hubble Constant, and is denoted with a subscript 0, i.e.,

$$H_0 = \frac{\dot{a}_0}{a_0}$$

For practical reasons, H_0 is generally quoted in units of km/sec/Mpc, though obviously, this reduces to the inverse time units.

Redshift

The time a photon takes to travel a distance $d\ell$ is $dt = d\ell/c$. During this time, the scale factor of the universe will increase by $da = \dot{a}\, dt$. The relative speed of a galaxy due to expansion is therefore

$$dv = H d\ell = \frac{\dot{a}}{a} c\, dt = c\frac{da}{a}$$

Now let's define redshift, z, as

$$z = \frac{\lambda_{\rm observed} - \lambda_{\rm emitted}}{\lambda_{\rm emitted}} = \frac{\Delta\lambda}{\lambda}$$

For small velocities, $dv \approx c\, z$, and $d\lambda = \Delta\lambda$, so substituting this in

$$dv = c\frac{da}{a} = cz = c\frac{d\lambda}{\lambda} \quad \Longrightarrow \quad \frac{da}{a} = \frac{d\lambda}{\lambda}$$

In other words, the fractional change in the wavelength of the photon is the same as the fractional change in the size of the universe.

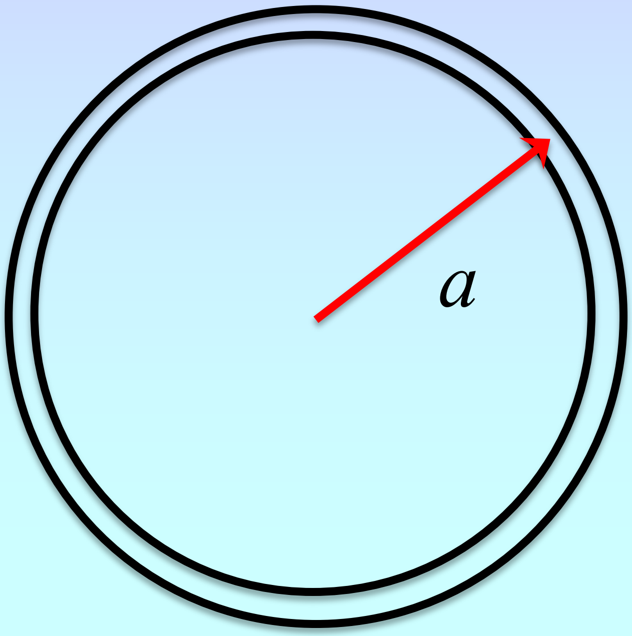
Redshift

The previous calculation was for photons traveling only a small distance. But one can then repeat the process, and thus integrate the photon's path over any distance. The result is the same: the fractional change in wavelength is equivalent to the fractional change in the size of the universe.

$$(1 + z) = \frac{a_0}{a}$$

The redshift defines the relative size of the universe at the time the photon was emitted relative to the size today.

Dynamics of a Newtonian Universe



Let's pick a center to a Newtonian universe, and consider the gravitational acceleration on a shell of material at distance a from the center. (a can be the extreme limit of the universe, or if you wish, it can be just a small region.)

$$\ddot{a} = -\frac{GM}{a^2}$$

If we multiply both sides by $\dot{a}$ and integrate over time

$$\int \dot{a}\,\ddot{a}\,dt = -\int \frac{GM}{a^2}\dot{a}\,dt$$

Both sides are of the form $\int u\, du$, so doing the integrals is easy

$$\frac{1}{2}\dot{a}^2 - \frac{GM}{a} = E$$

Dynamics of a Newtonian Universe

$$\frac{1}{2}\dot{a}^2 - \frac{GM}{a} = E$$

Obviously, E, the total energy comes from the constants of integration. If the potential energy of the universe is greater than its kinetic energy, $E < 0$, and eventually there will be a collapse. If the kinetic energy is greater than the potential energy, $E > 0$.

Of course, the total "mass" of the universe is difficult or impossible to measure. But we can substitute density using

$$M = \frac{4}{3}\pi a^3 \rho$$

If we multiply the energy equation by $2/a^2$ and make this substitution, then

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8}{3}\pi G\rho = \frac{2E}{a^2}$$

The Age of the Universe

Now let's calculate the age of the universe. Begin with

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8}{3}\pi G\rho = \frac{2E}{a^2}$$

Since the mass of the universe isn't changing, but the size is, the density of the universe today is related to the density in the past by

$$\rho_0 = \rho\left(\frac{a_0}{a}\right)^3$$

So the equation for the motion of the universe is

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8}{3}\pi G\rho_0\left(\frac{a_0}{a}\right)^3 = \frac{2E}{a^2}$$

Now let's solve it.

An Empty (Milne) Universe

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8}{3}\pi G\rho_0 \left(\frac{a_0}{a}\right)^3 = \frac{2E}{a^2}$$

If the universe is empty, then $\rho_0 = 0$, so

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{2E}{a^2} \implies \dot{a} = (2E)^{1/2}$$

E is a constant, so this is trivially integrated over time so that

$$t = \frac{a}{(2E)^{1/2}}$$

or, since $\dot{a} = (2\,E)^{1/2}$,

$$t = \frac{a}{\dot{a}} = \frac{1}{H}$$

and the age of the universe today would be $t_0 = 1/H_0$.

An Empty (Milne) Universe

Also note that in a Milne universe, $\dot{a} = (2\,E)^{1/2}$ is a constant. So

$$t = \frac{a}{\dot{a}} \quad \text{and} \quad t_0 = \frac{a_0}{\dot{a}_0} \quad \Longrightarrow \quad \frac{t}{t_0} = \frac{a}{a_0}$$

But recall that by the definition of redshift

$$(1+z) = \frac{a_0}{a}$$

so the age of the universe at any redshift is

$$\frac{t}{t_0} = \frac{a}{a_0} = (1+z)^{-1} \quad \Longrightarrow \quad t = \frac{t_0}{(1+z)} = \frac{1}{H_0}\frac{1}{(1+z)}$$

and the look-back time is

$$\Delta t = t_0 - t = \frac{1}{H_0}\left\{1 - \frac{1}{1+z}\right\}$$

A Critical (Einstein-de Sitter) Universe

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8}{3}\pi G\rho_0 \left(\frac{a_0}{a}\right)^3 = \frac{2E}{a^2}$$

For a critical universe, $E = 0$, so

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8}{3}\pi G\rho_0 \left(\frac{a_0}{a}\right)^3 = 0 \implies \dot{a} = \left\{\frac{8}{3}\pi G\rho_0 a_0^3\right\}^{1/2} a^{-1/2}$$

This differential equation actually has an analytical solution, with

$$a = C\,t^{2/3} \quad \text{with} \quad C = \left(6\pi G\rho_0 a_0^3\right)^{1/3}$$

(Try it!) The age of the universe is then

$$\left(\frac{\dot{a}}{a}\right) = H = \frac{(2/3)C\,t^{-1/3}}{C\,t^{2/3}} = \frac{2}{3}\frac{1}{t}$$

and the age of the universe today would be two-thirds $1/H_0$.

A Critical (Einstein-de Sitter) Universe

Now let's look at how redshift translates to time

$$(1+z) = \frac{a_0}{a} = \frac{C\, t_0^{2/3}}{C\, t^{2/3}} = \left(\frac{t_0}{t}\right)^{2/3}$$

So the age of the universe at any redshift is

$$\left(\frac{t}{t_0}\right)^{2/3} = \frac{1}{1+z} \implies t = t_0 \left(\frac{1}{1+z}\right)^{3/2} = \frac{2}{3}\frac{1}{H_0}\left(\frac{1}{1+z}\right)^{3/2}$$

and the look-back time is

$$\Delta t = t_0 - t = \frac{2}{3}\frac{1}{H_0}\left\{1 - (1+z)^{-3/2}\right\}$$

This is different from the Milne universe. The translation of redshift into age depends on the cosmology.

A Critical Density of the Universe

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8}{3}\pi G\rho = \frac{2E}{a^2}$$

For a critical universe, $E = 0$, so the critical density at any time is

$$\frac{8}{3}\pi G\rho_c = H^2 \quad \Longrightarrow \quad \rho_c = \frac{3H^2}{8\pi G}$$

For the universe today, this works out to $1.9 \times 10^{-29}\, h^2$ g/cm^3, where the variable $h = H_0/100$ km/s/Mpc.

One can then define a dimensionless density by scaling to this number, i.e.,

$$\Omega = \frac{\rho}{\rho_c} \quad \text{which for today is} \quad \Omega_0 = \frac{\rho_0}{\rho_{c,0}}$$

A General Universe

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8}{3}\pi G\rho_0 \left(\frac{a_0}{a}\right)^3 = \frac{2E}{a^2}$$

If $E = 0$, the solution to this equation is $a = C\ t^{2/3}$. If $E \neq 0$, the solution to the equation is still analytic, but it cannot be written as simply $a(t)$. Instead, one must define a dummy variable θ, and use

$\underline{E < 0}$	$\underline{E > 0}$
$a(\theta) = B(1 - \cos\theta)$	$a(\theta) = B(\cosh\theta - 1)$
$t(\theta) = \frac{B}{c}(\theta - \sin\theta)$	$t(\theta) = \frac{B}{c}(\sinh\theta - \theta)$

with $B = \frac{4\pi G\rho_0}{3c^2} a_0^3$. So, for example, $\dot{a}$ for a closed universe is

$$\dot{a} = \frac{da}{dt} = \frac{B\sin\theta\, d\theta}{\frac{B}{c}(1 - \cos\theta)d\theta} = \frac{c\sin\theta}{(1 - \cos\theta)}$$

Foreshadowing

Note that everything we have done up to now is based solely on Newton's view of the universe, and the assumption that the only force acting in the universe is gravity. We can release the latter assumption quite easily. If, for example, there was another force at work applying pressure (p) to the shell, then instead of

$$\ddot{a} = -\frac{GM}{a^2} = -\frac{4}{3}\pi G\rho\, a$$

the acceleration term would be

$$\ddot{a} = -\frac{4}{3}\pi G\,(\rho + 3p)a$$

The inclusion of this component would create an extra term on the right hand side of the equation of motion.

Einstein's Universe

Using Newton's laws, we derived

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8}{3}\pi G\rho_0\left(\frac{a_0}{a}\right)^3 = \frac{2E}{a^2}$$

Under general relatively, the equation is

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8}{3}\pi G\rho_0\left(\frac{a_0}{a}\right)^3 = -\frac{kc^2}{a^2} + \frac{\Lambda}{3}$$

The two new terms are

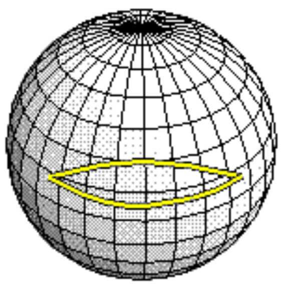
- k : This is the curvature of the universe. Since k only appears with the scale factor a (which is not a measurable quantity), it takes only 3 values: +1 (positive curvature, like a sphere), –1 (negative curvature, like a saddle), and 0 (flat space)
- Λ : An additional force, like a pressure. Expressed this way, it is a Cosmological Constant.

Curved Spacetime

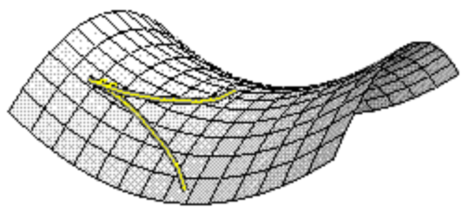
If $k = 0$: space is flat (like paper)

$k = +1$: space is positively curved (like a ball)

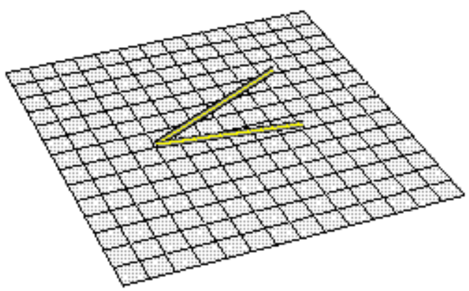
$k = -1$: space is negatively curved (like a saddle)



A *closed* universe curves "back on itself". Lines that were diverging apart come back together. Density > critical density.



An *open* universe curves "away from itself". Diverging lines curve at increasing angles away from each other. Density < critical density.



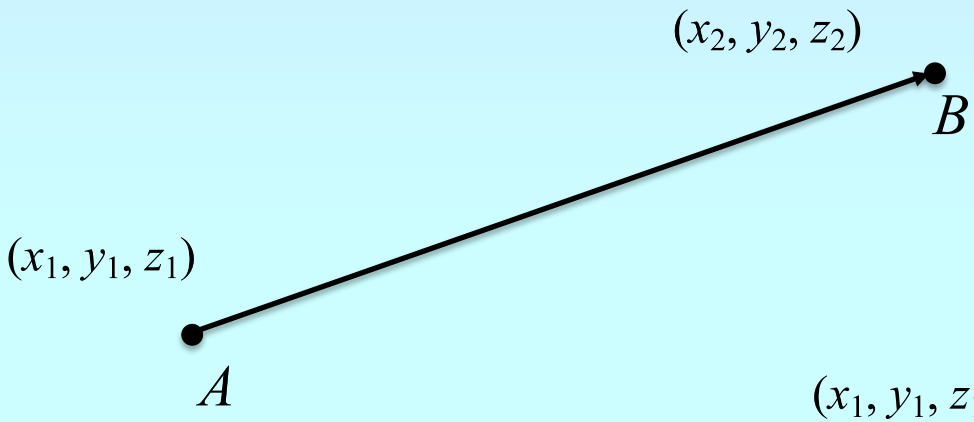
A *flat* universe has no curvature. Diverging lines remain at a constant angle with respect to each other. Density = critical density.

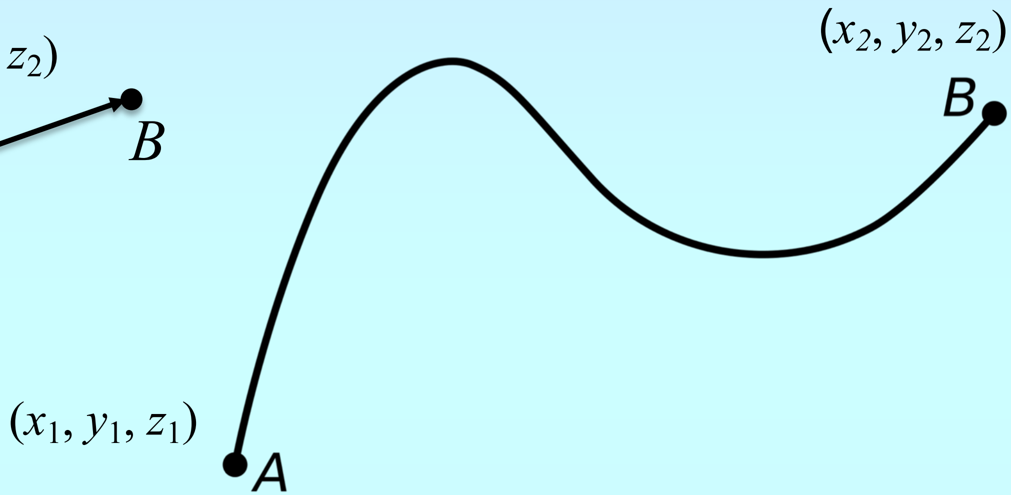
This means that distances and angles don't necessarily obey the rules learned in high-school geometry class.

$k = 0$	Triangle = 180°
$k = +1$	Triangle > 180°
$k = -1$	Triangle < 180°

Cosmological Paths

How do you calculate the distance from Point A to Point B?





The only way to measure the length of the curvy line is to integrate along the curve, i.e., calculate dx, dy, and dz at each location along the path. Measuring cosmological distances is similar – the path may be curvy (for $k = \pm 1$), *and* the space is constantly getting bigger.

Geometry

When measuring path lengths in cosmology, there are several complications:

- We are dealing with space-time. Thus, the path integration must include both space and time

$$ds^2 = c^2 dt^2 - du^2$$

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$$du^2 = d\xi^2 + \xi^2 d\theta^2 + \xi^2 \sin^2\theta \, d\phi^2$$

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- The universe is constantly expanding. So the space coordinate is constantly changing

$$du^2 = a^2 \left\{ d\xi^2 + \xi^2 \theta^2 + \xi^2 \sin^2 \theta \, d\phi^2 \right\}$$

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- Cartesian coordinates are non-optical. Since the universe is isotropic, we should use spherical coordinates and put the Earth at the origin
- The universe is constantly expanding. So the space coordinate is constantly changing
- Space may not be flat. The above metric is only applicable for flat space. On a sphere (or a saddle), the distance between any two points is

$$du^2 = \frac{d\xi^2}{1 - k\xi^2} + \xi^2 d\theta^2 + \xi^2 \sin^2\theta \, d\phi^2$$

The Robertson-Walker Metric

In an expanding universe, the separation between any two points in space-time therefore is

$$\begin{aligned} ds^2 &= c^2 dt^2 - a^2 du^2 \\ &= c^2 dt^2 - a^2 \left\{ \frac{d\xi^2}{1 - k\xi^2} + \xi^2 d\theta^2 + \xi^2 \sin^2\theta \, d\phi^2 \right\} \end{aligned}$$

This is the Robertson-Walker metric. But note that the equation is not as bad as it seems, as it usually can be simplified. For example, if we are observing a distant object, we can define ourselves to be at the origin; the light path will then be purely radial, and $d\theta = d\varphi = 0$.

Also, note that since we are often dealing with a light ray, the separation in space-time $ds^2 = 0$.

Proper Distance

Let's try an example: calculating how far light has traveled from redshift z to today. We start with the Robertson-Walker metric.

$$ds^2 = c^2 dt^2 - a^2 \left\{ \frac{d\xi^2}{1 - k\xi^2} + \xi^2 d\theta^2 + \xi^2 \sin^2\theta \, d\phi^2 \right\}$$

We choose our coordinate system such that the path is purely radial. Since we are dealing with light, $ds^2 = 0$, so the metric simplifies to

$$ds^2 = c^2 dt^2 - a^2 \left\{ \frac{d\xi^2}{1 - k\xi^2} \right\} = 0$$

If we take the square root and move the scale-factor a to the left-hand size of the equation

$$\frac{c}{a} dt = \frac{d\xi}{(1 - k\xi^2)^{1/2}}$$

Proper Distance

The proper distance between a point in the universe at space-time coordinates (u, t_1) and today $(0, t_2)$ is thus

$$\int_{t_0}^{t_1} \frac{c}{a} dt = \int_0^u \frac{d\xi}{(1-k\xi^2)^{1/2}}$$

For simplicity, let's compute this integral for an Einstein-de Sitter universe, i.e., one with $k = 0$. In this case, the right-hand integral is trivial. Also, as we have seen for a critical universe, $a = C\, t^{2/3}$. So

$$\frac{3c}{C}\, t_0^{1/3} - \frac{3c}{C}\, t_1^{1/3} = u$$

Now recall our definition of redshift

$$(1+z) = \frac{a_0}{a} = \frac{Ct_0^{2/3}}{Ct_1^{2/3}} = \left(\frac{t_0}{t_1}\right)^{2/3} \quad \Longrightarrow \quad t_1 = t_0(1+z)^{-3/2}$$

Proper Distance

If we use this ratio to substitute for t_1, then

$$u = \frac{3c}{C} t_0^{1/3} \left\{ 1 - (1+z)^{-1/2} \right\}$$

Finally, to get rid of the constant of proportionality, C, let's use another substitution, this time $\dot{a} = \{2/3\}\ C\ t^{-1/3}$. Then

$$u = \frac{2c}{\dot{a}_0} \left\{ 1 - (1+z)^{-1/2} \right\}$$

Now the distance is the co-moving coordinate u times the size of the universe, a. For the present day universe, $D_p = a_0 u$. So

$$D_p = a_0 u = \frac{a_0}{\dot{a}_0} 2c \left\{ 1 - (1+z)^{-1/2} \right\} = \frac{2c}{H_0} \left\{ 1 - (1+z)^{-1/2} \right\}$$

This is called the "proper distance" to a galaxy.

Proper Distance for Small z

$$D_p = \frac{2c}{H_0}\left\{1 - (1+z)^{-1/2}\right\}$$

Note that at small redshifts, this simplifies. If one expands the expression using the first 2 terms of a Taylor series about $z = 0$, then

$$f(z) = \left\{1 - (1+z)^{-1/2}\right\}\Big|_{z=0} = 0$$

$$f'(z) = \frac{1}{2}(1+z)^{-3/2}\Big|_{z=0} = \frac{1}{2}$$

$$D_p \approx \frac{2c}{H_0}\left\{f(z)\Big|_{z=0} + f'(z)\Big|_{z=0} z\right\} \approx \frac{cz}{H_0}$$

Since $cz \approx v$, the distance reduces to the well-known $v \approx H_0 D_p$.

Proper Distance

The previous calculation was for a critical $k = 0$ universe. The calculation for $k \neq 0$ (with $\Lambda = 0$) is slightly more complicated, but still analytic. (Try it sometime!) In that case

$$D_p = \frac{2c}{H_0 \Omega_0^2 (1+z)} \left\{ \Omega_0 z + (\Omega_0 - 2) \left[(\Omega_0 z + 1)^{1/2} - 1 \right] \right\}$$

And if you like that, you'll love the general expression for the age of the universe:

$$t_0 = \frac{1}{H_0} \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}} \left[\cos^{-1}\left(\frac{2}{\Omega_0} - 1 \right) - \frac{2}{\Omega_0} (\Omega_0 - 1)^{1/2} \right] \quad \text{for} \quad \Omega_0 > 1$$

$$t_0 = \frac{1}{H_0} \frac{\Omega_0}{2(1 - \Omega_0)^{3/2}} \left[\frac{2}{\Omega_0} (1 - \Omega_0)^{1/2} - \cosh^{-1}\left(\frac{2}{\Omega_0} - 1 \right) \right] \quad \text{for} \quad \Omega_0 < 1$$

You do have the pieces to solve this – it's tedious, but straightforward!

Obtaining a World Model

We have just seen that:

$$D_p = f(z, H_0, \Omega_0)$$

$$\Delta t = f(z, H_0, \Omega_0)$$

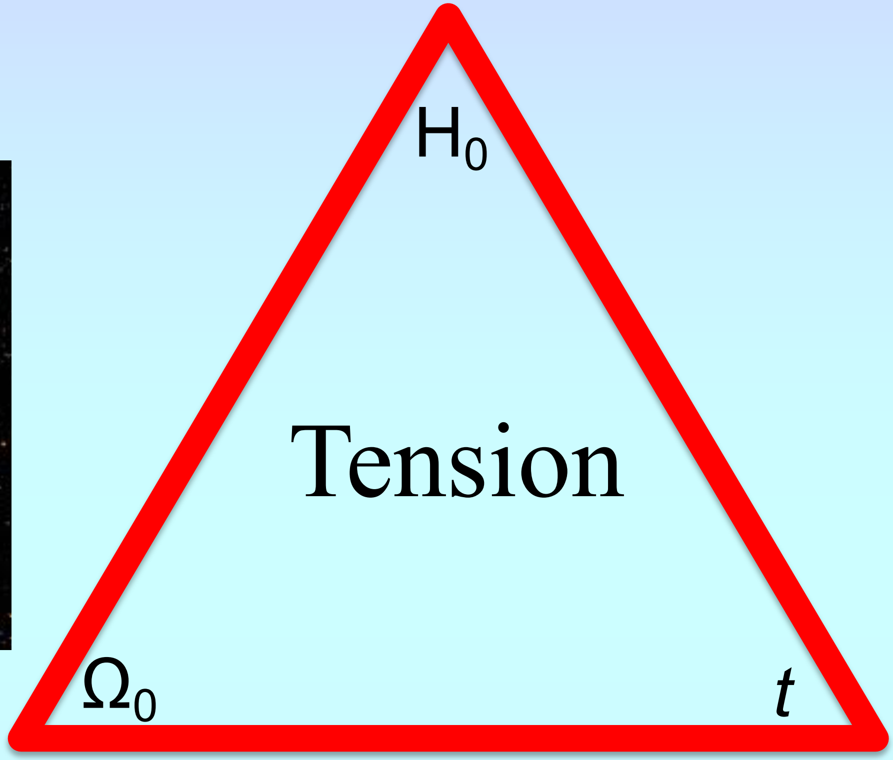
$$t_0 = f(H_0, \Omega_0)$$

So if we could measure the distance to a set of objects, we could translate those distances into constraints on the cosmological world model. Similarly, if we could place limits on the age of the universe, we would be constraining the cosmological parameters.

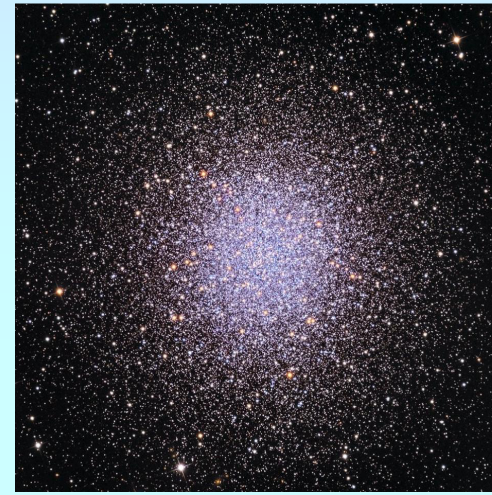
Surveys suggest
$H_0 \gtrsim 75$ km/s/Mpc

Matter Density of the Universe

Surveys suggest
$\Omega_M \sim 0.2$



Age of the Oldest Stars



Measurements suggest $t \sim 13$ Gyr

The “Hubble Wars” of the 1980’s and 1990’s were largely about the tension between measurements of the Galaxy’s oldest stars, the Hubble Constant, and matter density. There did not seem to be a consistent solution.

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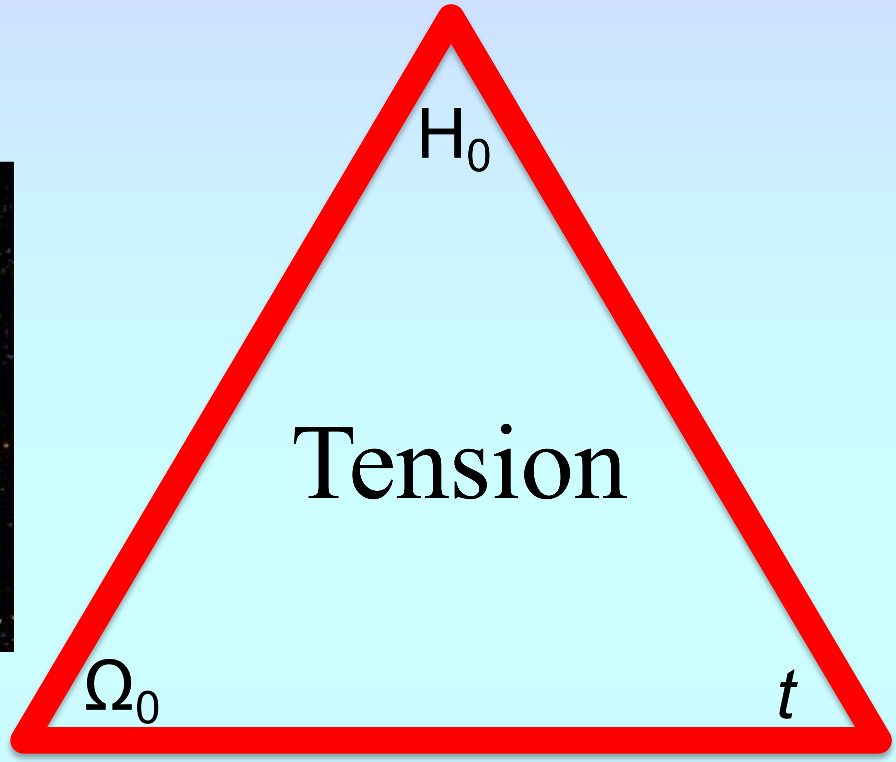

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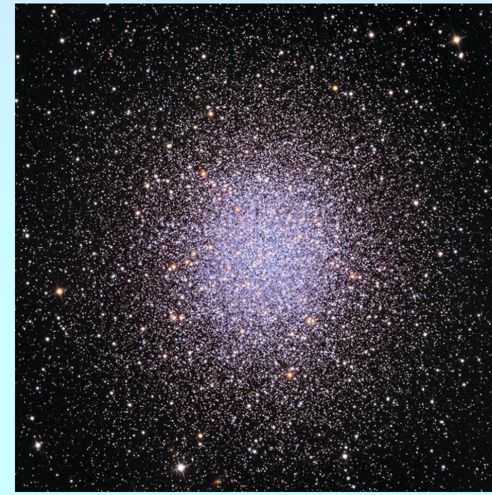


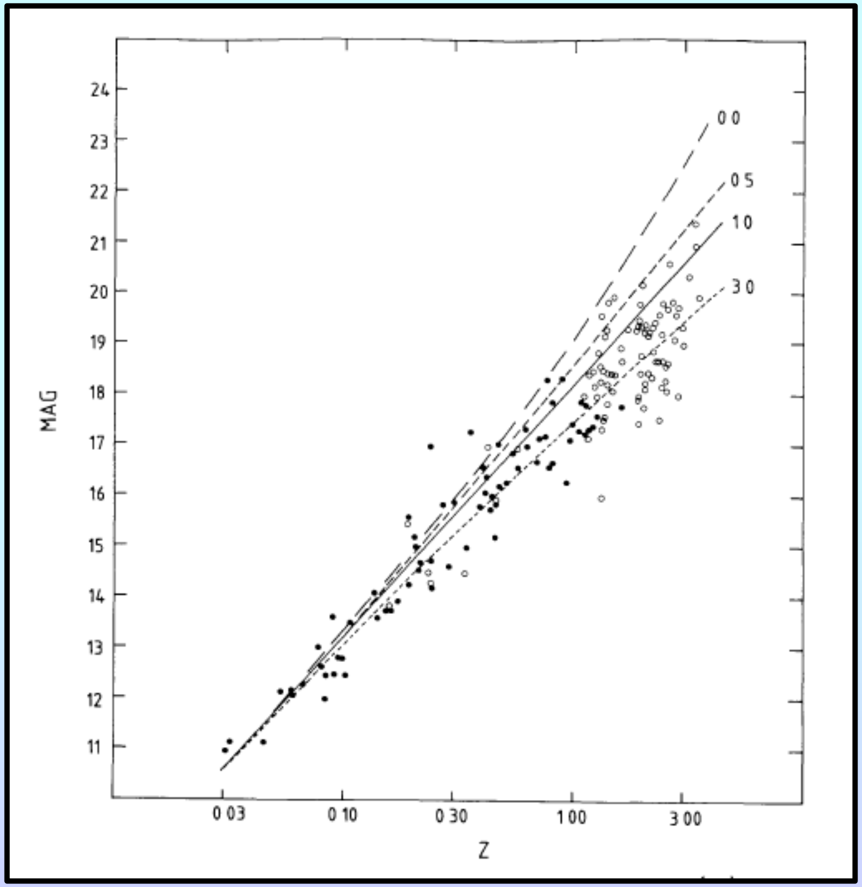
Measurements suggest $t \sim 13$ Gyr

H_0 Value	Result
50 km/s/Mpc	All measurements consistent
75 km/s/Mpc	Only works if the universe is rather empty
100 km/s/Mpc	Stars are too old, even with an empty universe
Theorists preferred Ω_0 = 1	

1998: The Accelerating Universe

The action of gravity over the age of the universe should have slowed down the cosmological expansion. This slowing is especially important early on, when the universe was small and dense. To quantify this effect, astronomers defined a dimensionless deceleration parameter:

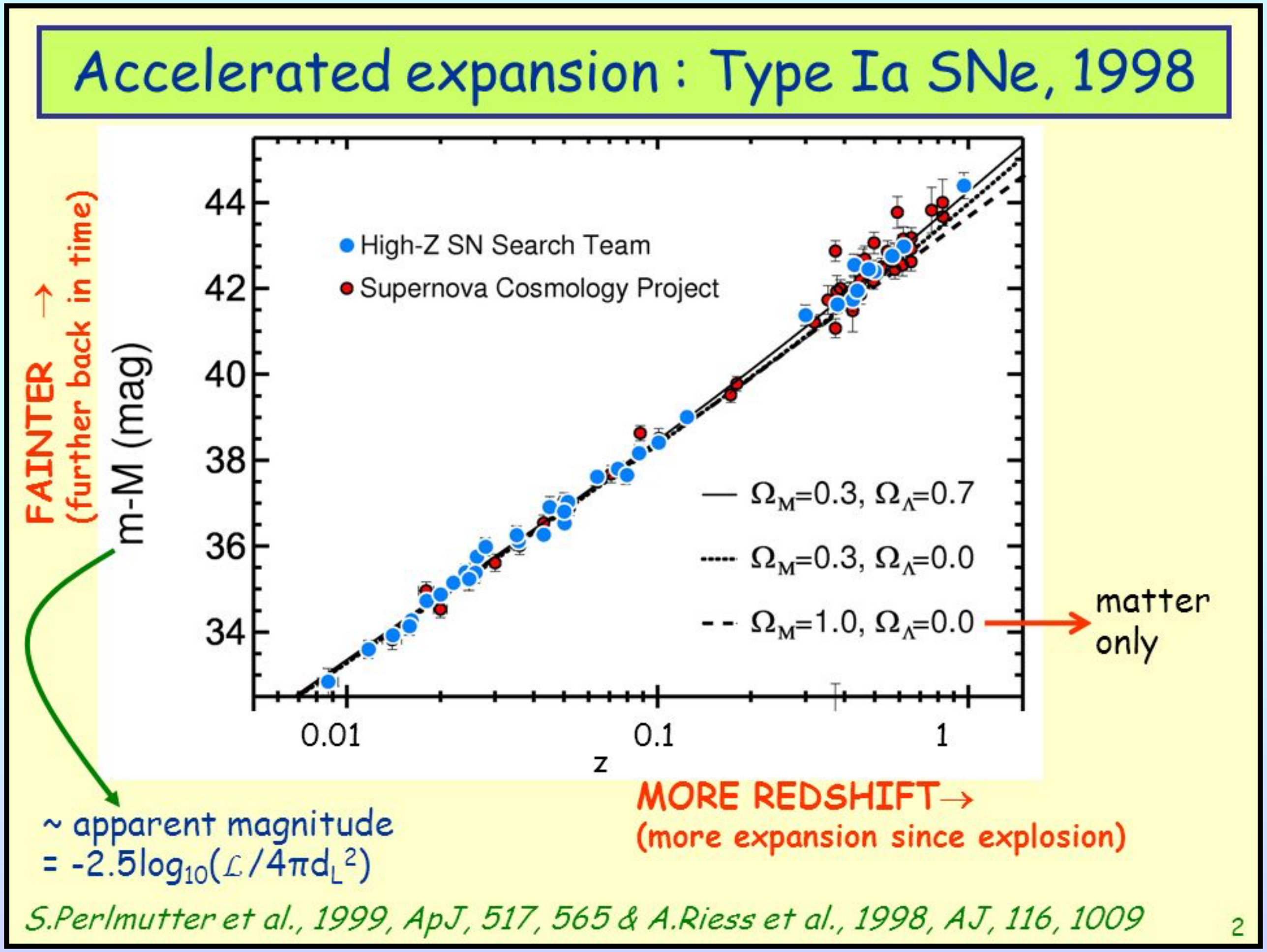
$$q = -\frac{\ddot{a}a}{\dot{a}^2}$$



For a universe governed entirely by gravity, $q = \Omega / 2$. Before 1998, cosmology was viewed in this context, with the measurement of q_0 and H_0 being a major goal. For cosmologies with high q (or Ω), the greater deceleration produced distances that were shorter.

1998: The Accelerating Universe

In 1998, everything changed. Working independently (but with some overlap in the supernova datasets), the two groups came up with the same result: the expansion wasn't slowing down: it was accelerating!!!



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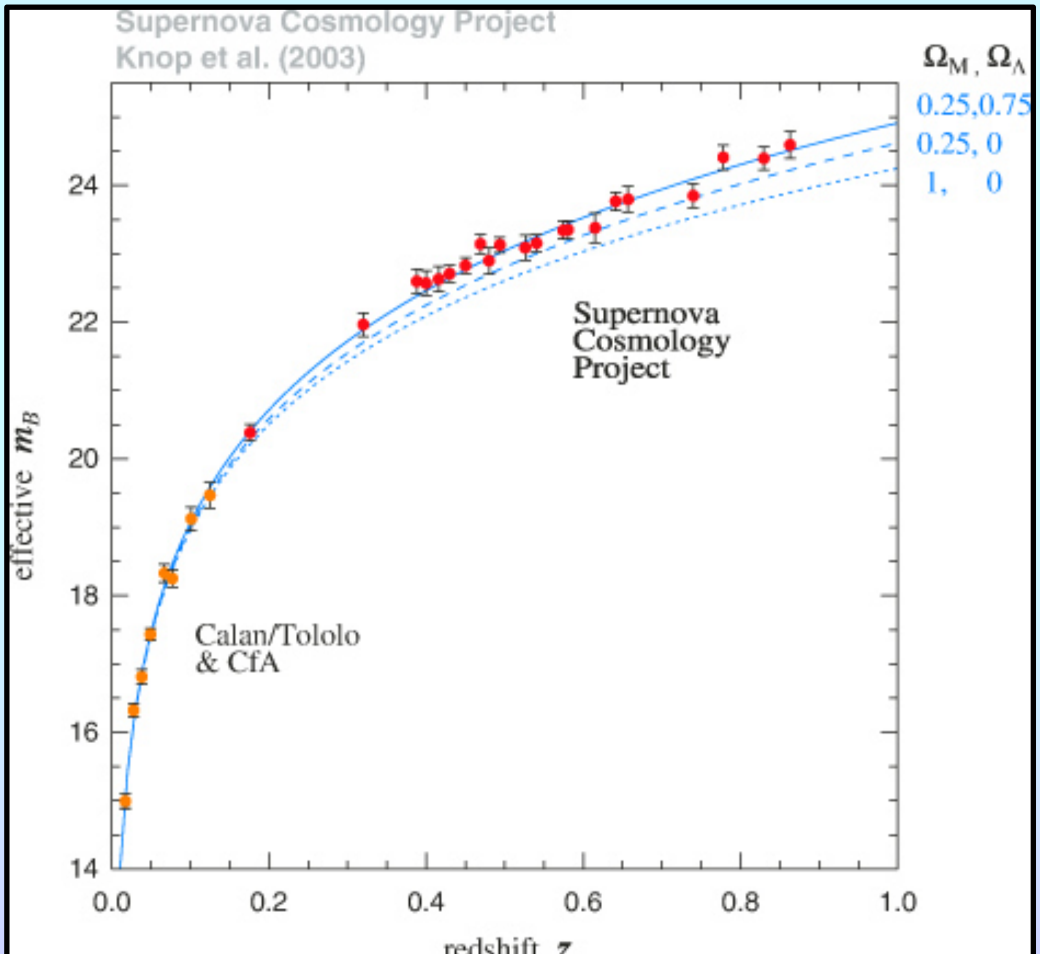


The simplest explanation is to invoke the Cosmological Constant, which Einstein inserted into his equations to keep the universe static.

This cause of this acceleration has been termed "Dark Energy." But note that Dark Energy is not necessarily a Cosmological Constant.

Cosmology with a Cosmological Constant

What does the Cosmological Constant do? It provides an additional potential to the universe, which manifests itself as repulsive pressure like force,

$$U_\Lambda = -\frac{1}{6}\Lambda c^2 R^2 \quad \Longrightarrow \quad \vec{F}_\Lambda = -\frac{dU}{da}\hat{a} = \frac{1}{3}\Lambda c^2 a\,\hat{a}$$

Note that the "force" is directly proportional to the scale factor a of the universe. The larger the universe, the greater the repulsive force. (Small note here: the nomenclature for Λ is inconsistent: some papers incorporate the c^2 term into Λ, some do not.)

This means that in the early universe, cosmic acceleration was unimportant, as the effect doesn't become measurable until $z \lesssim 4$. But the acceleration increases with a. This solves the tension of the 1990s: because $H(z)$ was smaller in the past, it took longer for the universe to get to its present size.

Cosmology with a Cosmological Constant

For a $\Lambda = 0$ universe, the critical density was easily-defined; it was the density which provided the universe with just enough gravity to halt the expansion at time of infinity: an $E = 0$ (or $k = 0$) universe.

$$\rho_c = \frac{3}{8\pi G} H^2$$

With a Cosmological Constant, things are different. For example to keep $k = 0$, the critical density of the universe must be

$$\rho_c = \frac{3H^2 - \Lambda c^2}{8\pi G}$$

Cosmology with a Cosmological Constant

Because "density" is no longer a well-defined quantity, what is commonly done is to keep the old definition of ρ_c, and define three new dimensionless quantities: one for matter, one for the energy density of the Cosmological Constant, and one for curvature:

$$\Omega_M = \frac{\rho}{\rho_c} = \frac{8\pi G}{3H_0}\rho_c$$

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2}$$

$$\Omega_k = -\frac{k}{a_0^2 H_0^2}$$

$$q = \frac{1}{2}\Omega_M - \Omega_\Lambda$$

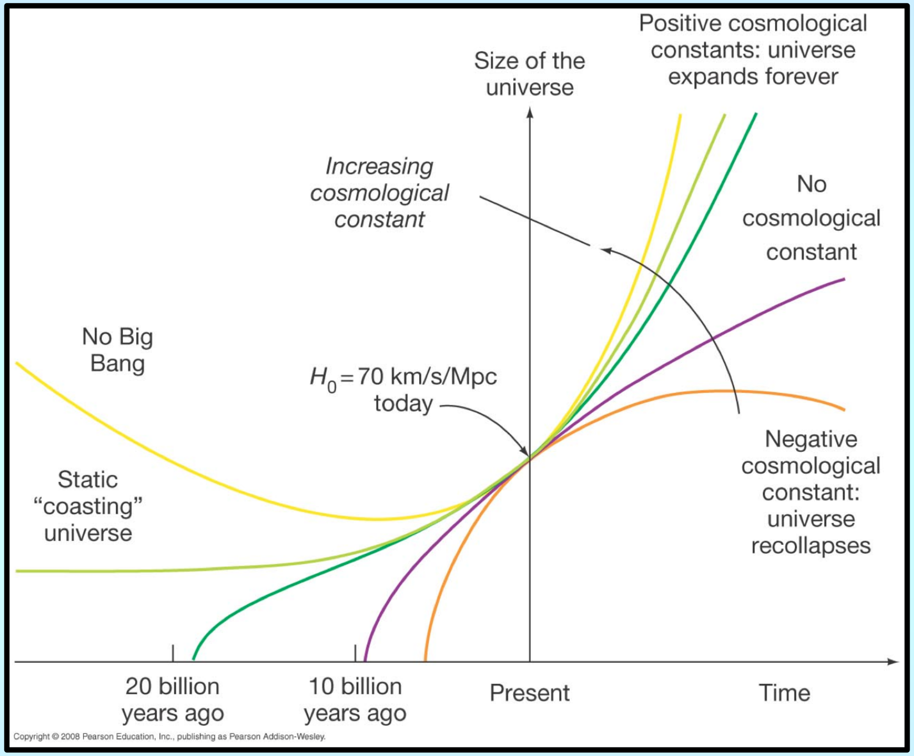
Using these definitions,
$\Omega_M + \Omega_\Lambda + \Omega_k = 1$

In an inflationary universe,
$\Omega_k = 0$, and $\Omega_M + \Omega_\Lambda = 1$

Cosmology with a Cosmological Constant

With Λ, even universes which expand forever can be spatially flat.

Note that since the dark energy increases as $F_\Lambda \sim a$ while gravitational attraction decreases as $F_{\text{grav}} \sim 1/a^2$, as the universe expands, Λ becomes more and more important. In the limit, the cosmological constant dominates, with $H = (\Lambda/3)^{1/2}$ and $a \propto \exp(H\,t)$.



Cosmology with a Cosmological Constant

When we include a non-zero Cosmological Constant, the solution to the Friedmann equation is no longer analytic.

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8}{3}\pi G\rho_0 \left(\frac{a_0}{a}\right)^3 = -\frac{kc^2}{a^2} + \frac{\Lambda}{3}$$

With the previous definitions and letting $x = a_0/a$, the Friedmann equation can be re-written as

$$\left(\frac{1}{H_0}\right)^2 \left(\frac{dx}{dt}\right)^2 = 1 + \Omega_M \left(\frac{1}{x} - 1\right) + \Omega_\Lambda \left(x^2 - 1\right)$$

This equation, and hence the look-back time and proper distance, must be computed numerically. The look-back time as a function of redshift becomes

$$\Delta t = \frac{1}{H_0}\int_0^z (1+z)^{-1}\left\{(1+z)^2(1+\Omega_M z) - \Omega_\Lambda z(2+z)\right\}^{-\frac{1}{2}} dz$$

Cosmology with a Cosmological Constant

Similarly, the equation for proper distance is

$$
D_p = \begin{cases} \dfrac{c}{H_0}|\Omega_k|^{-1/2}\sinh\left\{|\Omega_k|^{1/2}E\right\} & \Omega_k < 0 \\[2ex] \dfrac{c}{H_0}E & \Omega_k = 0 \\[2ex] \dfrac{c}{H_0}\Omega_k^{-1/2}\sin\left\{\Omega_k^{1/2}E\right\} & \Omega_k > 0 \end{cases}
$$

where $E = \displaystyle\int_0^z \left\{(1+z)^2(1+\Omega_M z) - \Omega_\Lambda z(2+z)\right\}^{-\frac{1}{2}}$

In other words

$$D_p = f(z, \Omega_M, \Omega_\Lambda)$$

$$\Delta t = f(z, \Omega_M, \Omega_\Lambda)$$

$$t_0 = f(z, \Omega_M, \Omega_\Lambda)$$

Cosmology with a Cosmological Constant

A number of publicly available programs are available to perform cosmological calculations with a Cosmological Constant. These include the python code cosmocalc, the free *CosmoCalc* program for an *iphone,* and Ned Wright's on-line calculator at

http://www.astro.ucla.edu/~wright/CosmoCalc.html

“Best” Current Values

The current “best” values (from the Planck satellite) for the densities of the universe are:

$$H_0 = 67.37 \pm 0.54 \text{ km/s/Mpc}$$

$$\Omega_\mathrm{M} = 0.3147 \pm 0.0074$$

$$\Omega_\Lambda = 0.6847 \pm 0.0073$$

$$\Omega_\mathrm{k} = 0.001 \pm 0.002$$

$$t_0 = 13.797 \pm 0.0023 \text{ Gyr}$$

(so the universe is getting 2.18×10^{-18} times bigger every second.)

Beyond a Cosmological Constant

The motivations for modeling the cosmic acceleration with a Cosmological Constant are

- It's the simplest way of writing the equation
- Einstein included it in his formulation (to prevent a static universe from collapsing)

But there's no physics that says the Cosmological Constant must be a constant; maybe Λ changes with time or the scale factor of the universe. A more general formulation is to describe the effects of Λ as a pressure term.

Beyond a Cosmological Constant

Recall that a more general description of the acceleration on a shell of material was

$$\ddot{a} = -\frac{4}{3}\pi G\,(\rho + 3p)a$$

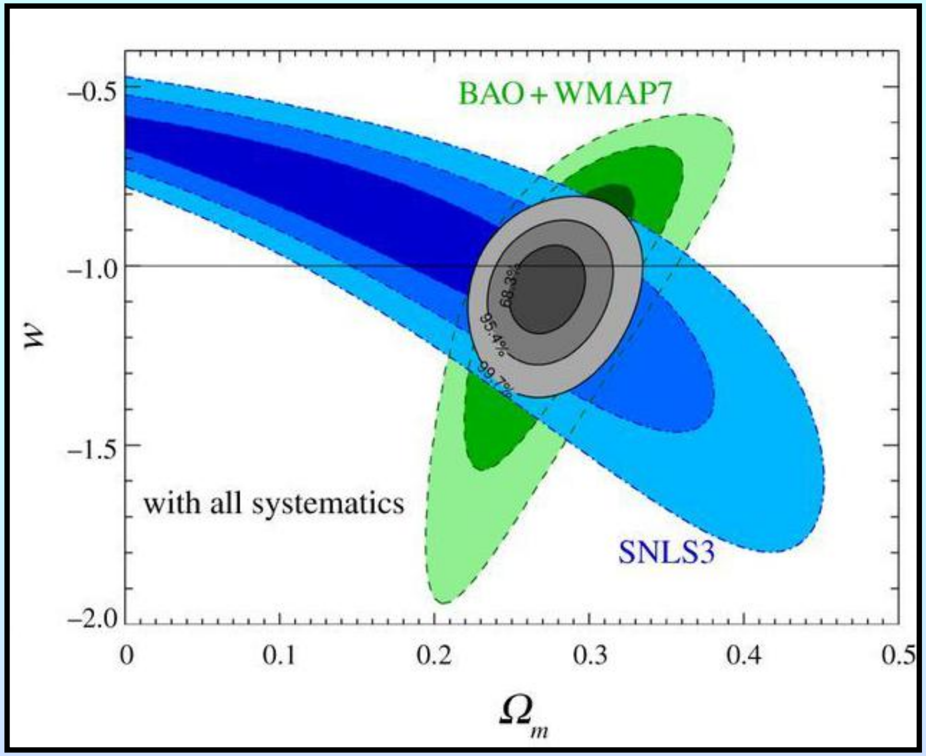
Suppose we now relate pressure to density (as is done with gas) with a cosmological "equation of state," $p = w\,\rho$

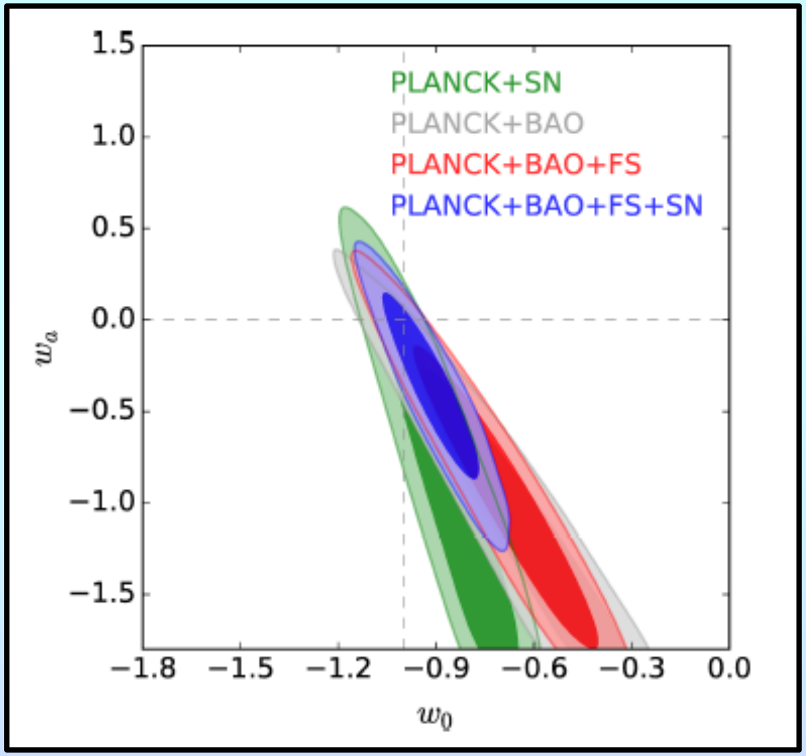
$$\ddot{a} = -\frac{4}{3}\pi G(1 + 3w)\rho\,a$$

For a universe with just gravity, $w = 0$. For a universe with a Cosmological Constant, $w = -1$. But if the acceleration is caused by more than a Cosmological Constant, then w will change with time. To first order, one can then write $w = w_0 + w_a\,(1 - a)$.

Beyond a Cosmological Constant

There are many theories for cosmic acceleration that go beyond the simple cosmological constant. To address them, one solves the Friedmann equation with the hypothesized term, and tries fitting to the data. The results are plots like this:





Luminosity Distance

We have previously computed the proper distance to an object, i.e., the path length. But if one wants to use the inverse square law of light to measure cosmological distances, there are other factors to consider. The observed flux one measures is

$$f = \frac{L}{4\pi D^2} = \frac{n \cdot h\nu_e}{dt_e} \; \frac{1}{4\pi D_p^2}$$

But the emitted frequency of the photon is not the observed frequency: due to redshift, $\nu_0 = \nu_e / (1+z)$. Also, $dt_0 \neq dt_1$: because the object is moving away from us, there is time dilation. In addition, because the source is moving away from us, the time between pulses gets longer, as the 2nd pulse has longer to travel. As a result,

$$dt_0 = dt_e(1 + z)$$

Luminosity Distance

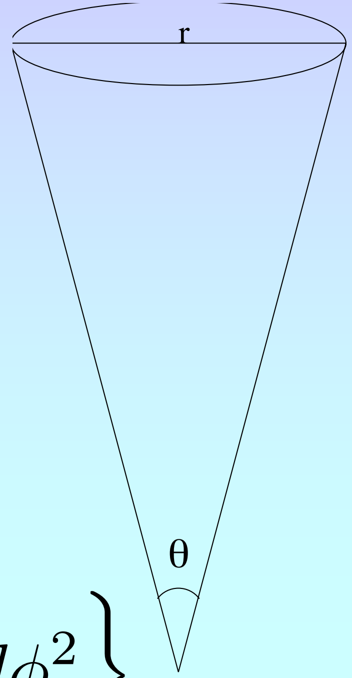
In other words, there are two extra (1+z) terms in the relationship between flux and luminosity: one from the loss of energy due to redshift and from time dilation. So

$$f = \frac{n \cdot h\nu_e}{dt_e} \; \frac{1}{4\pi D_p^2} = \frac{n \cdot h\nu_0}{dt_0} \; \frac{1}{4\pi D_p^2} \frac{1}{(1+z)^2}$$

We can therefore define "luminosity distance" as the distance one should use when applying the inverse square law. From the above equation, this distance is related to proper distance by

$$D_L = D_p\,(1+z)$$

Angular Size Distance



Now let's examine how an object's angular size changes with distance. Locally, of course, an object with size *s* will subtend an angle $\theta = s / d$.

To make this measurement, we start with the Robertson-Walker metric.

$$ds^2 = c^2 dt^2 - a^2 \left\{ \frac{d\xi^2}{1 - k\xi^2} + \xi^2 d\theta^2 + \xi^2 \sin^2\theta \, d\phi^2 \right\}$$

If the object is in the plane of the sky, then the radial coordinate to each side is the same, so $d\xi = 0$. Similarly, both sides of the object are being observed at the same time, so $dt = 0$. Finally, we can choose our coordinate system so that the angle subtended lies entirely in the –θ direction, so $d\varphi = 0$. So the metric is simply

$$ds^2 = a^2 du^2 = a^2 \xi^2 d\theta^2$$

Angular Size Distance

If we take the square root, then $\theta = s / a\xi$, or if we use $a = a_0 / (1+z)$

$$\theta = \frac{s}{a\xi} = \frac{s(1+z)}{a_0\xi}$$

But remember the definition of proper distance is $D_p = a_0\, u = a_0\, \xi$.

So

$$\theta = \frac{s(1+z)}{D_p}$$

This is the "angular diameter" distance, i.e., the distance one should apply when using a standard ruler. Obviously

$$D_A = D_p\,(1+z)^{-1}$$

Note the angular size of an object, $\theta \propto (1+z)$. At high redshift, an object of constant size will get larger, as it takes up a proportionally larger fraction of the universe!

Cosmological Distances

We now have three different cosmological distances

- The proper distance, or actual path length, D_p
- The luminosity distance for the inverse square law, D_L
- The angular diameter distance for size computations, D_A

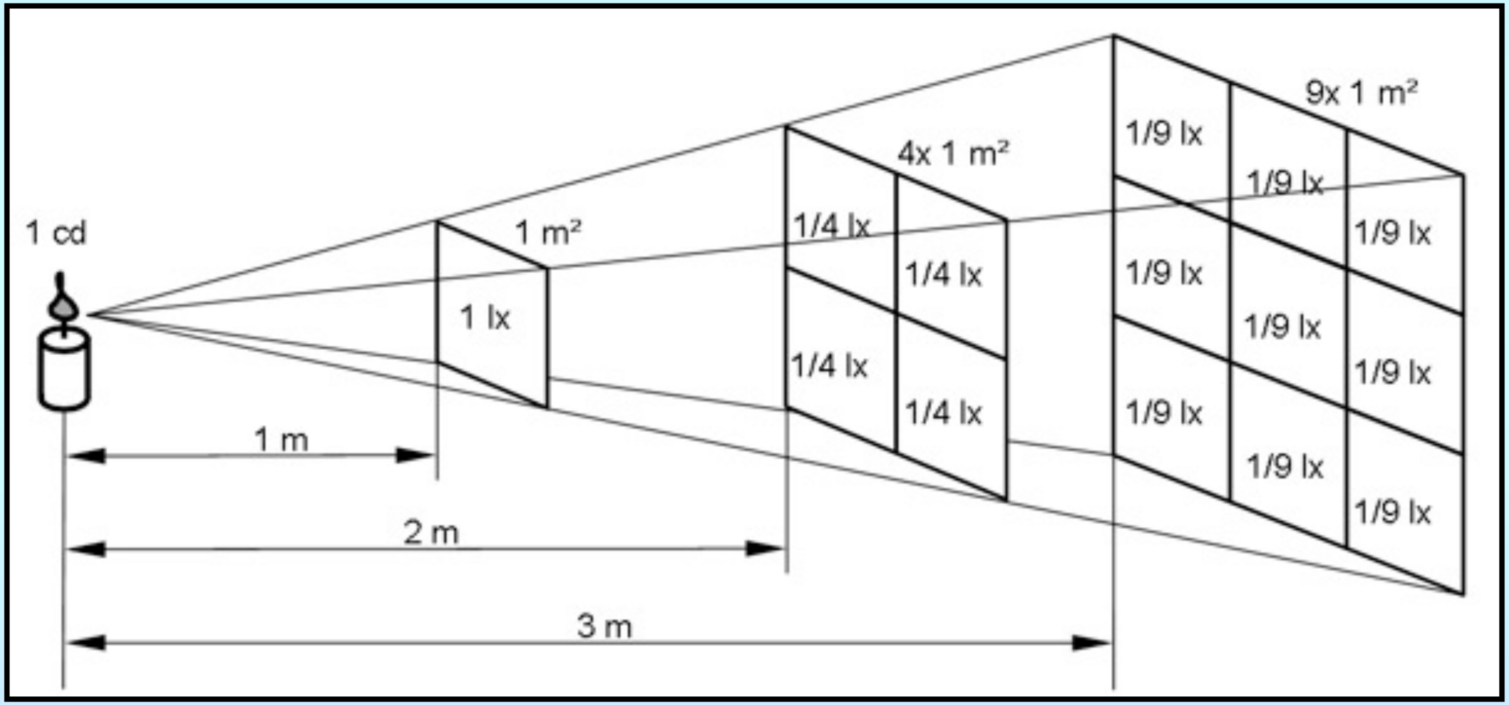
The relationship between these distances are

$$D_L = D_p\,(1 + z) \qquad D_A = D_p\,(1 + z)^{-1} \qquad D_A = D_L\,(1 + z)^{-2}$$

Now let's see what happens when these distances collide.

Surface Brightness

Recall that surface brightness, Σ, is normally independent of distance: as distance increases, the flux per star decreases by $1/r^2$, but the area increases by r^2. So light per unit area remains the same.



However, let's look at this relation using the equations of cosmology.

Surface Brightness

Surface brightness is luminosity per area, so

$$\Sigma = \frac{f}{\theta^2} = \frac{L}{4\pi D_L^2} \cdot \frac{D_A^2}{s^2}$$

$$= \frac{L}{4\pi D_p^2 (1+z)^2} \cdot \frac{D_P^2}{s^2 (1+z)^2}$$

$$= \frac{L}{4\pi s^2} (1+z)^{-4}$$

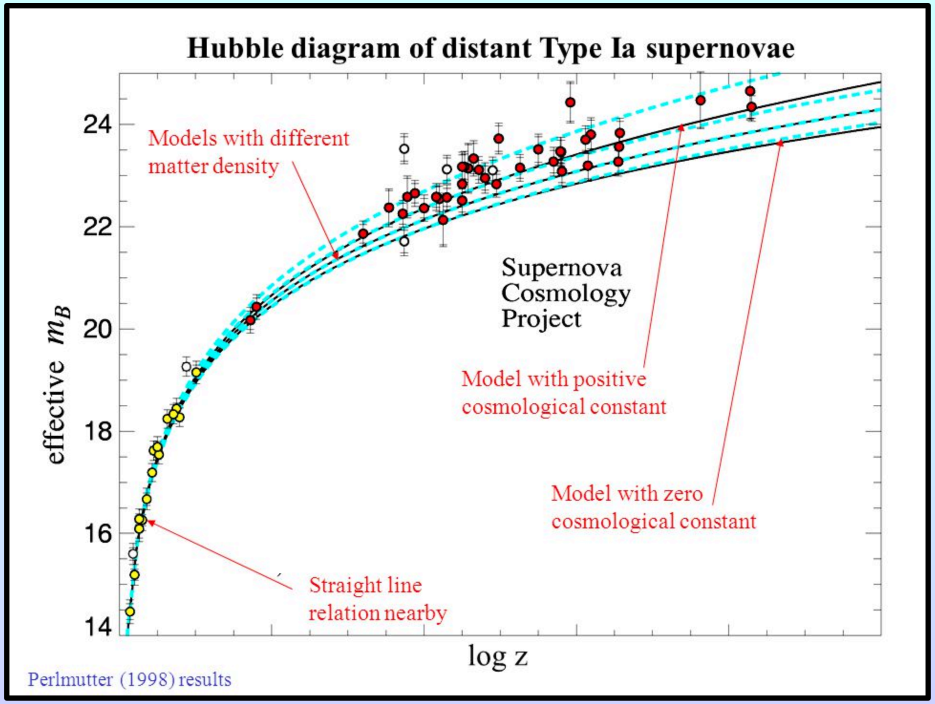
$$= \Sigma_e (1+z)^{-4}$$

Cosmological surface brightness dimming is important! At $z = 4$, the light per unit area will be 625 times less than normal. That's almost 7 magnitudes!

Summary

As the expression for D_p shows, the distance to an object is a function of the cosmology, i.e., D_p (and therefore D_L and D_A) is a function of $f(z, H_0, \Omega_m, \Omega_\Lambda)$. Which means that

- The sizes and luminosities that one infers depend on the cosmology.
- If one knows an object's luminosity and/or size ahead of time, one can invert the problem and determine the cosmology.



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